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FLOW OF NON-NEWTONIAN COUPLE STRESS FLUID BETWEEN TWO MOVING PARALLEL DISKS: EXACT SOLUTIONS AND STABILITY ANALYIS

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Abstract- In this work the Non-Newtonian couple stress fluid flow between two moving parallel disks is considered. The exact solutions to the steady Navier-Stokes equations for the incompressible Non-Newtonian couple stress fluid flow and stability due to disks moving towards each other or in opposite directions with a constant velocity. The graph of the stationary solutions is depicted. The motion near stagnation point, the periodical one-dimensional perturbation is applied and found that the movement of disks determines the stability of the solution. Stationary solutions in the form of jets are studied.

1. INTRODUCTION

Stokes [1] has formulated the theory of couple-stress fluid. One of the applications of couple-stress fluid is its use in study of mechanisms of lubrication of Synovial fluids. The synovial fluid has been modeled as a couple-stress fluid in human joints by Walicki and Walicka [7].

There is a large class of processes which can be considered from the mathematical point of view as the motion of liquid between two parallel disks, moving towards each other or in opposite directions with a constant velocity. These include such processes as the motion of liquid through a hydraulic pump, and the motion of underground water can also be described with a help of the current model. In fig.1, these two applications are presented. It should be noted that in spite of different types of hydro dynamical problem at first sight, the mathematical descriptions are the same. So it is possible to describe the water motion in a hydraulic pump (when impermeable disks are moving toward or apart each other) similarly to the motion of underground water (when permeable disks are fixed). The second case refers to water motion through porous media. These problems are interesting because some of their solution analytically obtained, can be confirmed by experiments.

Suppose, two parallel disks placed in water and start moving towards each other or in opposite directions, assuming the size of the disks to be much larger than the distance between them. Even with a qualitative assessment we can see that when the disks are approaching each other the effort required is smaller than that for separation when the disks are moving apart. This can be explained by the different character of the liquid motion. When the disks are approaching it is potential: when the disks are moving apart it is rotational.

This process deals with a description of the types of possible instability of such motion. Craik & Criminale [2] described a procedure for finding class of exact solutions of the Navier-Stokes equations. These solutions consist of a ‘basic flow’ with spatially uniform rates of strain and a ‘disturbance’ of a planar form: the disturbance is continuously distorted by the basic flow but nevertheless remains of planar form at all times. This is similar to the formulation, given by Lagnado, Phan-Thien & Leal (1984), but was restricted to two-dimensional basic flows and the authors were unnoticed that their linearized approximation is in fact an exact solution for single plane-wave modes.

The aims of this chapter are presented in two parts. The first is to generalize the results of Craik [3] in a case of plane-wave superposition. The second is to find the possible forms of the jet solutions which are generated as a result of the instability development.

2. MATHEMATICAL FORMULATION

Consider the motion of viscous incompressible couple stress fluid induced by two parallel disks moving towards each other, in this case where $h << l$ (where h is the distance between the disks and l is the length of the disks). Let us assume that the horizontal velocity does not depend on the vertical coordinate, whereas the vertical velocity depends linearly on the distance between the disks.

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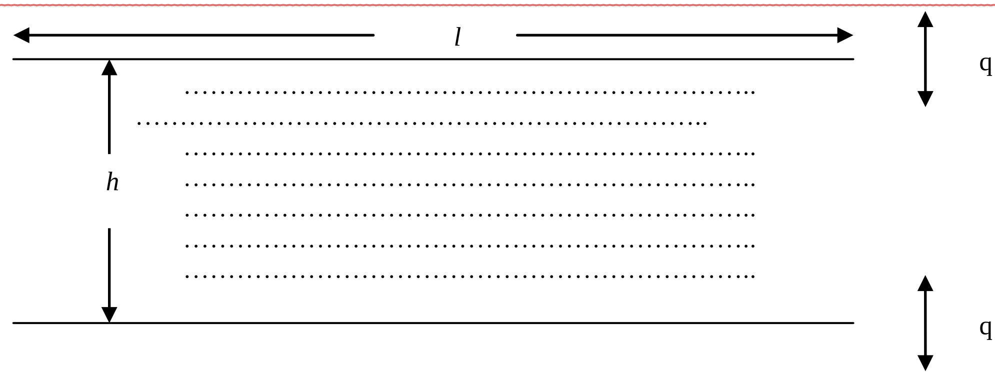

Figure: 1 The parallel disks are moving to and fro with the velocity q

In this case the Navier-Stokes equations have the following form

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=2q \tag{2.1}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{\partial p}{\partial x}+\nu\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right]-\eta\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right]^2 u \tag{2.2}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{\partial p}{\partial y}+\nu\left[\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right]-\eta\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right]^2 v \tag{2.3}$$

where ν is the kinematic viscosity, η is the couple stress fluid parameter and the velocity components are represented as

$$u=u(x,y,t) \qquad v=v(x,y,t) \qquad w=-2qz \qquad p=p(x,y,t)-2q^2z^2 \tag{2.4}$$

Therefore the above equation becomes

$$\frac{\partial \omega}{\partial t}+u\frac{\partial \omega}{\partial x}+\omega\frac{\partial u}{\partial x}+v\frac{\partial \omega}{\partial y}+\omega\frac{\partial v}{\partial y}=\nu\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right]\omega-\eta\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right]^2\omega \tag{2.5}$$

Where ω is the vorticity given by $\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}=\omega$

3. METHOD OF ANALYSIS

For an analysis let us consider the potential component from the horizontal components of the velocity and introduce the flow function

$$u=q\,x+\frac{\partial \psi}{\partial y} \tag{3.1}$$

$$v=q\,y-\frac{\partial \psi}{\partial x} \tag{3.2}$$

where ψ is the stream function. Now continuity equation (2.1) is satisfied identically and momentum equation (2.5), after elimination of the pressure and introduction of the vorticity $\omega=\nabla\times u$, will give the equations of motion in the following form:

$$\frac{\partial \omega}{\partial t}+\left\{qx+\frac{\partial \psi}{\partial y}\right\}\frac{\partial \omega}{\partial x}+\left\{q+\frac{\partial^2 \psi}{\partial x\partial y}\right\}\omega+\left\{qy-\frac{\partial \psi}{\partial x}\right\}\frac{\partial \omega}{\partial y}+\left\{q-\frac{\partial^2 \psi}{\partial x\partial y}\right\}\omega=\nu\Delta\omega-\eta\Delta^2\omega$$

where $\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$

$$\frac{\partial \omega}{\partial t}-\{\psi,\omega\}=-q\left[\frac{\partial}{\partial y}(y\omega)+\frac{\partial}{\partial x}(x\omega)\right]+\nu\Delta\omega-\eta\Delta^2\omega \tag{3.3}$$

where $\{\psi,\omega\}$ denotes the Poisson brackets:

$$\{\psi,\omega\}=\frac{\partial \psi}{\partial x}\frac{\partial \omega}{\partial y}-\frac{\partial \psi}{\partial y}\frac{\partial \omega}{\partial x} \tag{3.4}$$

One of the solution of the equation (3.3) is $\psi = 0$, which corresponds to liquid potential motion, known as the motion near the stagnation point (another solution for ψ is given in section IV). Following the work of Craik [3], to investigate the stability of this solution let us consider the periodical one-dimensional perturbation $\delta\psi$. This perturbation is expressed by the following equation.

$$\psi = \psi + \hat{\psi} = \psi + \delta\psi$$
$$\delta\psi = k^{-2}(t)A(t)\cos(k(t)x) \tag{3.5}$$

To analyze the change of the vorticity in the course of time we put the stream function $\delta\psi$ (3.5) into (3.3). That is

$$\psi = \delta\psi = k^{-2}(t)A(t)\cos(k(t)x)$$
$$\omega = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\psi = -A(t)\cos(k(t)x)$$

Comparing the coefficients with same x-powers to obtain the following system of nonlinear equations:

$$\frac{dk}{dt} = -q\,k(t) \tag{3.6}$$

$$\frac{dA}{dt} = -2qA(t) - \nu k^2(t)A(t) - \eta k^4(t)A(t) \tag{3.7}$$

By equation (3.6), we get

$$k(t) = k(0)e^{-qt} \tag{3.8}$$

Using this in equation (3.7), we get

$$A(t) = A(0)\exp\left[-2qt + \left(-1 + e^{-2qt}\right)\nu\frac{k^2(0)}{2q} + \eta\frac{k^4(0)}{4q}\left(-1 + e^{-4qt}\right)\right] \tag{3.9}$$

where $k(0),\ A(0)$ are constants, determining the amplitude and wavelength at the initial point of time. The sign of q in equation (3.9) determines the stability of the solution $\psi = 0$. When $q > 0$ the solution is stable, the amplitude $A(t)$ is decreasing: otherwise the solution is unstable; the amplitude $A(t)$ is increasing however, for $q < 0$ the solution is unstable only until $t = \left(\frac{1}{-2q}\right)Ln\left|\frac{-\nu \pm \sqrt{\nu^2 - 8q\eta}}{2\eta k^2(0)}\right|$ after which the amplitude decreases rapidly, owing to dissipation.

4. RESULTS AND DISCUSSIONS

4.1 Stability analysis

Let us consider the case when the flow function perturbation has the following form

$$\psi = \hat{\psi} + \delta\psi = \frac{A_1(t)}{k_1^{\,2}(t)}\cos[k_{11}(t)x + k_{12}(t)y] + \frac{A_2(t)}{k_2^{\,2}(t)}\cos[k_{21}(t)x + k_{22}(t)y] \tag{4.1}$$

We put the stream function $\delta\psi$ and comparing the coefficients of $\cos(k_{11}x + k_{12}y),\ \cos(k_{21}x + k_{22}y),$ $x\sin(k_{11}x + k_{12}y)$, $x\sin(k_{21}x + k_{22}y)$ $y\sin(k_{11}x + k_{12}y)$ & $y\sin(k_{21}x + k_{22}y)$ from equation (3.3) we get

$$\frac{dA_1}{dt} = -\left[2q + (\nu + \eta k_1^2)k_1^2\right]A_1 \tag{4.2}$$

$$\frac{dA_2}{dt} = -\left[2q + (\nu + \eta k_2^2)k_2^2\right]A_2 \tag{4.3}$$

$$\frac{dk_{11}}{dt} = -qk_{11} \qquad \Rightarrow\ k_{11}(t) = k_{11}(0)e^{-qt} \tag{4.4}$$

$$\frac{dk_{21}}{dt} = -qk_{21} \qquad \Rightarrow\ k_{21}(t) = k_{21}(0)e^{-qt} \tag{4.5}$$

$$\frac{dk_{12}}{dt} = -qk_{12} \qquad \Rightarrow\ k_{12}(t) = k_{12}(0)e^{-qt} \tag{4.6}$$

$$\frac{dk_{22}}{dt} = -qk_{22} \qquad \Rightarrow\ k_{22}(t) = k_{22}(0)e^{-qt} \tag{4.7}$$

Now $k_1^2 = k_{11}^2 + k_{12}^2 = \left\{k_{11}^2(0) + k_{12}^2(0)\right\} e^{-2qt} = k_1^2(0) e^{-2qt},\ k_1^4 = k_1^4(0) e^{-4qt},\ k_2^2 = k_2^2(0) e^{-2qt},\ k_2^4 = k_2^4(0) e^{-4qt},$ (4.8)

Substituting these in the equation (4.2) we get

$$\frac{dA_1}{A_1} = -\left[2q + \nu k_1^2(0) e^{-2qt} + \eta k_1^4(0) e^{-4qt}\right] dt$$

Solving this we get

$$A_1(t) = A_1(0) \exp\left[-2qt + \frac{\nu k_1^2(0)}{2q}(e^{-2qt} - 1) + \frac{\eta k_1^4(0)}{4q}(e^{-4qt} - 1)\right] \tag{4.9}$$

$$A_2(t) = A_2(0) \exp\left[-2qt + \frac{\nu k_2^2(0)}{2q}(e^{-2qt} - 1) + \frac{\eta k_2^4(0)}{4q}(e^{-4qt} - 1)\right] \tag{4.10}$$

In general,

Let us consider the case when the flow function perturbation has the following form

$$\delta\psi = \sum_{i=1}^{N} \frac{A_i(t)}{k_i^2(t)} \cos(k_{i1}(t)x + k_{i2}(t)y) \tag{4.11}$$

Provided that $k_{i1}^2 + k_{i2}^2 = k^2 \qquad \forall i$

For $q > 0,$ the solution is stable, with both the amplitude $A(t)$ and the wave number $k(t)$ decreasing in the course of time. And for $q < 0,$ the solution is unstable. However the amplitude increases until

$$t = \sum_{i=1}^{N} \left(-\frac{1}{2q}\right) Ln \left|\frac{q}{\nu k^2(0) + \eta k^4(0)}\right|$$

After which owing to dissipation it decreases rapidly. The wave number $k(t)$ increases in the course of time. The new and interesting fact which has been discovered in the course of this research is that the wave number $k(t)$, corresponding to the time $t = \sum_{i=1}^{N} \left(-\frac{1}{2q}\right) Ln \left|\frac{q}{\nu k^2(0) + \eta k^4(0)}\right|$ is not dependent on the initial conditions and is equal to $k = \sqrt{\frac{-\nu \pm \sqrt{\nu^2 + 4\eta q}}{2\eta}}$.

It should be noted that in each of the cases investigated $q > 0$ corresponds to the situation when the disks are moving towards each other and $q < 0$ to the situation when the disks are moving apart.

Note that if in equations (4.10) & (4.11) with $N = 1$, the results obtained by Craik [3] are retrieved. This case corresponds to a perturbation in a form of one plane wave. The case when $N > 1$ corresponds to plane-wave superposition, which can (for special conditions for wave number and amplitude (Chandrasekhar 1997)) reduce to the appearance of different space structures.

Figures 2, 3, 4 and 5 give the variations of amplitude with respect to time as follows

(i) If the disks are moving apart the incompressible couple stress fluid is unstable up to the certain time, but the amplitude value of instable will reduces by the increase of couple stress parameter (η). If the disks are moving towards each other the couple stress fluid is stable with both amplitude and wave number at all time.

(ii) With the increase of kinematic viscosity υ, in the case of disks moving apart the amplitude value of instability decreases. This is due to the increase in the amplitude and wave number.

(iii) In the case of increase in the velocity of the disks in moving apart, the amplitude value of instable will also increases.

(iv) But in the case of the disks are moving towards each other, the flow is stable with the variation of all the parameters.

4.2 Stationary solutions in the form of jets

The solution $\psi = 0$, corresponding to the liquid motion near a stagnation point, has been considered. We also find and examine other situations of stationary solutions, such as jets. Consider the flow function in the following form:

$$\psi = xF(y) + \varphi(y) \tag{4.12}$$

Substituting equation (4.12) into equation (3.3) it takes the following form

$$\psi_x \omega_y - \psi_y \omega_x = -q(2\omega + y\omega_y + x\omega_x) + \nu\Delta\omega - \eta\Delta^2\omega \tag{4.13}$$

And since

$$\omega = xF''(y) + \varphi''(y) \tag{4.14}$$

Equation (4.13) can be rewritten as

$$F[xF''' + \varphi'''] - [xF' + \varphi']F'' = -q[2xF'' + 2\varphi'' + xyF''' + y\varphi''' + xF''] + \nu[xF'^{v} + \varphi'^{v}] - K_c[xF^{vi} + \varphi^{vi}] \tag{4.15}$$

where a prime denotes a derivative with respect to the argument (here y) and a superscripts denotes the derivatives. Equating groups of terms with the same x powers it is possible to obtain the following system:

$$FF''' - F'F'' = -q[3F'' + yF'''] + \nu F'^{v} - \eta F^{vi} \tag{4.16}$$

$$F\varphi''' - \varphi' F'' = -q[2\varphi'' + y\varphi'''] + \nu\varphi'^{v} - \eta\varphi^{vi} \tag{4.17}$$

We consider the particular solution of equation (4.16) as $F = a\,y$. In this case (4.17) will take the following form:

$$ay\varphi''' + q(2\varphi'' + y\varphi''') = \nu\varphi'^{v} - \eta\varphi^{vi} \tag{4.18}$$

After some mathematical transformations and integrating twice, we obtain the following equation.

$$-\eta\varphi'^{v} + \nu\varphi'' = (a+q)y\varphi' - 2a\varphi \tag{4.19}$$

Equation (4.19) is a fourth order differential equation, for $\eta = 0$ the differential equation has the form of Hermit's differential equation when two conditions are satisfied: $\frac{a+q}{\nu} = 2$ and $\frac{a}{q}$ is non-negative integer. The solutions of this equation have the following form:

$$\varphi = \frac{d^n}{dy^n}\left(A\exp\left(\frac{q}{(3+n)\nu}y^2\right)\right) \tag{4.20}$$

where the relation between a and q is

$$a = -\frac{1+n}{3+n}q, \quad n \in [0, \infty] \tag{4.21}$$

Thus the solution of equation (3.3) can be written as

$$\psi = -\frac{1+n}{3+n}qxy + \frac{d^n}{dx^n}\left(A\exp\left(\frac{q}{(3+n)\nu}y^2\right)\right) \tag{4.22}$$

In equation (4.22), the first term denotes the liquid motion corresponding to the potential flow component and the second term denotes (represents) the jet behavior corresponds to non-potential flow component. When $q < 0, \quad n > 0, \quad \nu > 0,$ it is observed that, the second term approaches zero for $y \to \pm\infty$.

5. CONCLUSION

In this investigation, it is found that, if the disks are moving apart, the incompressible couple stress fluid is unstable up to certain time, and then it is stable after words. This is because of the wave number increases in the course of time. But in case of disks are moving towards each other the couple stress fluid is stable with both the amplitude and the wave number.

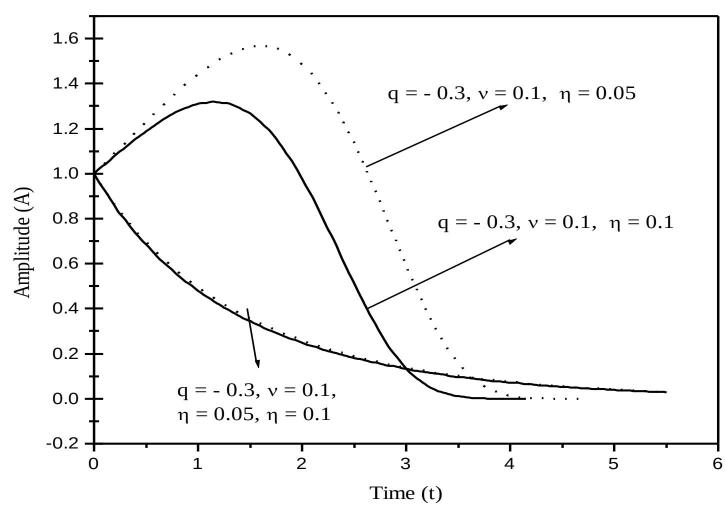

Fig. 2 Variations in amplitude $A(t)$ with time for different values of η and ν

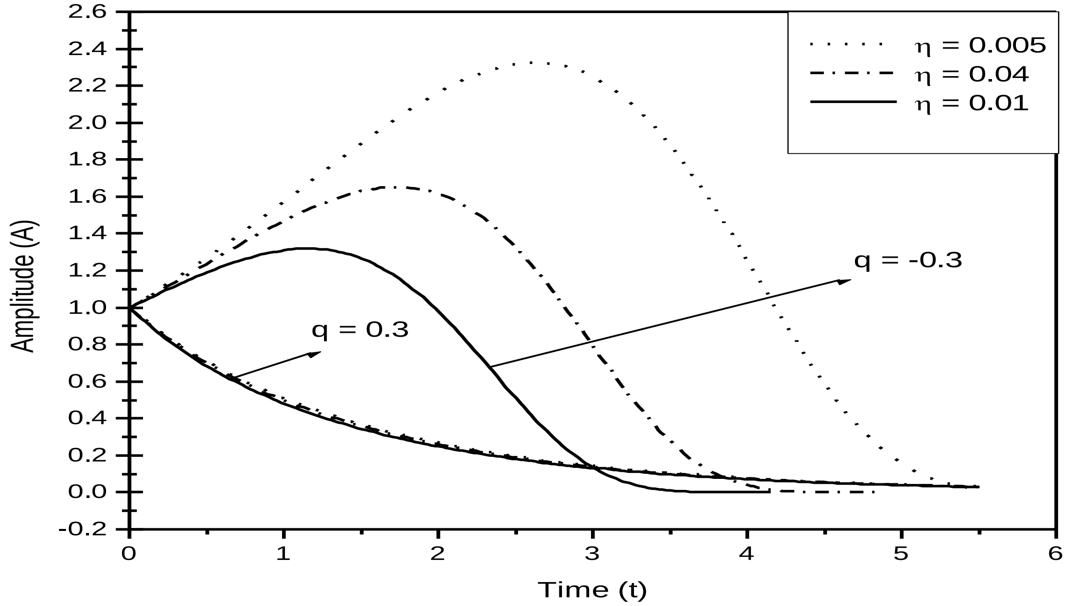

Fig. 3 Variations of η on the Amplitude $A(t)$ for $\upsilon = 0.1$

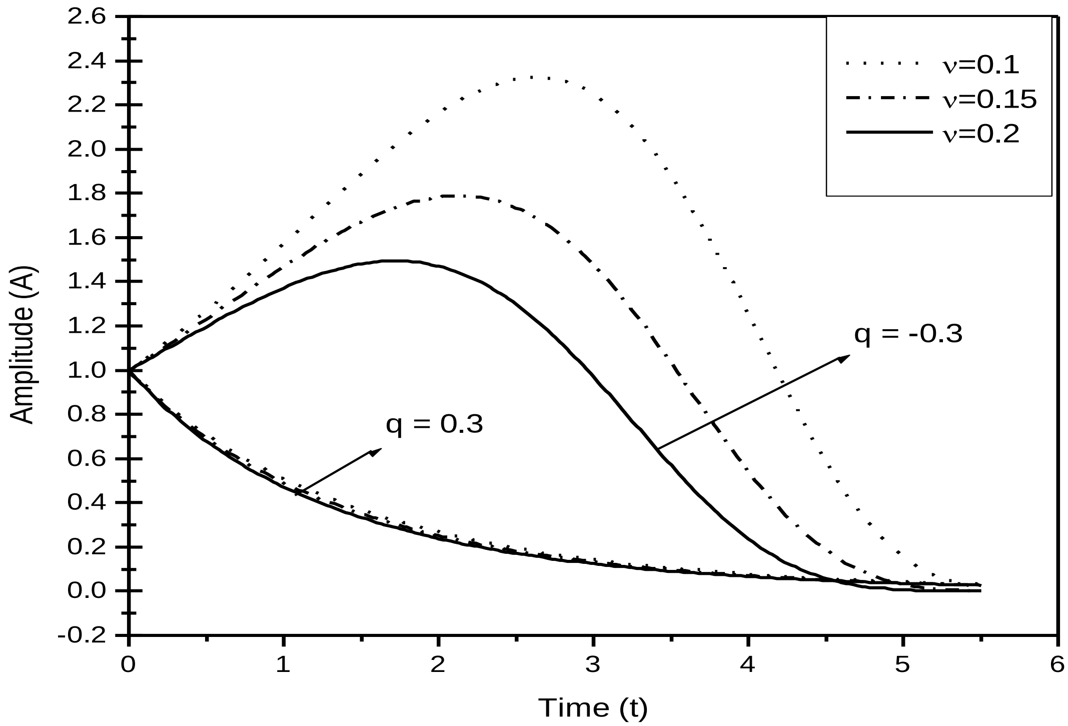

Fig. 4 The effect of υ with $\eta = 0.005$ on the Amplitude $A(t)$

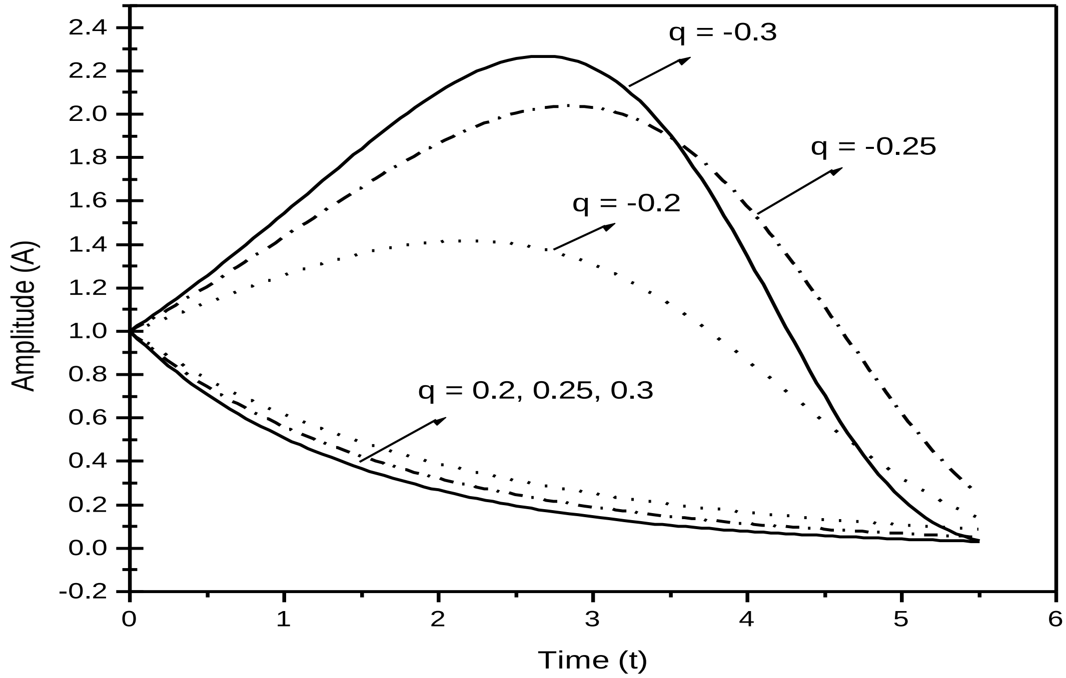

Fig. 5 Variations of q on the Amplitude $A(t)$ for $\eta = 0.005$ and $\upsilon = 0.1$

6. REFERENCES

[1] Stokes, V. K., 1966, Couple stresses in Fluids, Phys. Fluids vol.9, and pp.1709-1715.

[2] Craik, A., and Criminale, W., 1986, Evolution of wavelike disturbances in shear flows: a class of exact solutions of the Navier-Stokes equations, Proceedings of Royal Society London, vol.A 406, pp.13-36.

[3] Craik, A., 1989, The stability of unbounded two-and three-dimensional flows subject to body forces: some exact solutions, J. of Fluid Mech. vol.198, pp. 275-293.

[4] Balagondar, P M., and Kempe Gowda, M., Exact Solutions of Non-Newtonian Fluid Flow between Two Moving Parallel Disks and Stability Analysis, the International Journal of Applied Mathematical Sciences, vol.6, 2012, no. 37, pp.1827-1835.

[5] P. M. Balagondar., and M. Kempe Gowda., Exact solutions for the incompressible electrically conducting viscous flow between two moving parallel disks in unsteady magneto hydro dynamic and stability analysis, Int. J. of Applied Mechanics and Engineering, 2013, vol. 18, No. 2, pp. 571-579.

[6] Chandrasekhar, S., 1961, Hydrodynamic and hydromagnetic stability, Oxford University Press, London.

[7] Walicki, E. and Walicka, A. Interia effect in the squeeze film of a couple-stress fluids in biological bearings, Int. J. Appl, Mech. Engg., 1999, vol. 4, pp. 363-373.